Local Interactions of an Entangled State with the Environment via Amplitude Damping Channel: Some Consequences

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Abstract

Consider a two-qubit maximally entangled state that gets corrupted when *only one* of the qubits is subjected to local interaction with the environment via amplitude damping channel . We show there are situations where this corrupted state gets *better* by allowing the *other* qubit also to interact with the environment. Moreover if the corrupted state is non teleporting, then it can always be made teleporting by the same prescription i.e. by allowing the second qubit to interact with the environment.

Quantum entanglement [1] is a property of composite systems by virtue of which the local subsystems exhibit certain correlations among themselves that cannot be explained in classical terms. Often termed as quantum nonlocality, the subject had been under extensive study and debate since 1935 when Einstein, Podolsky and Rosen questioned the completeness of quantum mechanics [2]. Later, in 1964 in his pioneering work John Bell [3] showed that the correlations exhibited by the local subsystems that may be far apart are *inherently* quantum mechanical and any model admitting the description of local hidden variables (or one may prefer to say local realism) fails to explain such correlations.

The nonlocal correlations that exist between the parts of a physical system have been exploited in recent times for processing information [4] and to develop faster and efficient algorithms [5]. It is quantum entanglement, together with the quantum no-cloning principle [18] that makes quantum information so radically different from classical information. In particular, an entangled state acts as the channel for transmitting and distributing quantum information encoded in qubits. Thus we are allowed to send qubits from one place to another (Quantum Teleportation [6]) or to establish secret key among various distant parties (Quantum Key Distribution [7]) using only local quantum operations (e.g. a generalized measurement involving an ancilla) and exchanging surprisingly small amount of classical information. However for faithful quantum communication maximally entangled states are essential. For example, quantum teleportation [6] with fidelity one is achieved with certainty only with maximally entangled states although in real life preparing a perfect entangled state is a difficult problem. Even if we assume that there are no imperfections in preparation, still to protect such a perfectly prepared maximally entangled state from decoherence effects is exceedingly difficult.

One very important aspect in the study of entangled states is to obtain some litmus test for quantum inseparability. By this we mean that given a state how can we find out whether it is entangled or not? Unfortunately, only for $2 \otimes 2$ and $2 \otimes 3$ systems the necessary and sufficient conditions for entanglement has been obtained [8, 9]. In view of this, the notion of *fully entangled fraction* [10] turns out to be very useful specially in the case of bipartite systems. It is defined as,

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$$f(\rho) = max \langle \psi | \rho | \psi \rangle \tag{1}$$

Note that in the definition the maximization is carried out taking all maximally entangled states into account. It was shown in Refs. [11, 12] that given an entangled state, either pure or mixed, described by the density operator ρ, the *fully entangled fraction* as defined by (1) uniquely determines the teleportation fidelity. Thus the quantity $f(\rho)$ corresponding to a given state appropriately serves as a measure of the quality of the entangled state concerned i.e., the quality of any given entangled state is now judged by its efficiency to perform quantum teleportation. Specifically, a state ρ in the Hilbert space $C^d \otimes C^d$, is considered to be useful for teleportation if and only if $f(\rho) > 1/d$. Thus quantum teleportation [6] also serves as an operational way to *test* the presence and strength of quantum entanglement. Here we would like to mention that in this paper we will deal with only bipartite two qubit states and for those states the usefulness criteria is given by $f(\rho) > 1/2$. One should keep in mind that when we say that a state is "useful" for teleportation, we mean that, such a state when used for teleportation gives a better fidelity than what can be obtained classically.

As noted earlier, in practice an entangled state is susceptible to local interactions with environment resulting in a mixed entangled state or a separable state depending on the nature and strength of the interaction. Complete isolation of a quantum system from its local environment is impossible and destruction of the nonlocal correlations (either partially or fully) is inevitable. This is what is called decoherence. Such local dissipative interactions reduce the quality of the entangled state as a whole by reducing or destroying its capacity of information processing (for example, a reduction in teleportation fidelity). Now in general a qubit interacts with its local environment via the following channels [13]: depolarizing channel, phase damping channel and amplitude damping channel. Thus, given an entangled state, either one of the qubits or both (which are presumably far apart) can locally interact with the environment via any of these channels and invariably any such interaction associated with decoherence, results in a mixed entangled state or a separable state. If it so happens that a pure entangled state is transformed into a mixed entangled state then one can still generate a fewer number of maximally entangled states from an ensemble of mixed entangled states by applying the distillation protocols [14]. But one should also note that if such dissipative interactions turn an entangled state to a separable one, then of course one cannot distill out any entangled state by applying only local operations.

Although we understand that dissipative interactions with the local environment in general corrupts an entangled state but that this is not always true has been shown very recently by Badziag et. al. [15]. They have shown that local interaction with environment via *amplitude damping* can *enhance* fidelity of teleportation for a family of entangled states. In particular, a non teleporting state can be made teleporting when allowed to interact with environment via amplitude damping. This is indeed a very surprising result.

In this paper we report some novel consequences when a maximally entangled state is subjected to local interactions with the environment via the amplitude damping channel. Suppose we have a bipartite maximally entangled state, say, ρ and the qubits of this entangled pair undergo interactions with their respective local environments via amplitude damping. There are two possibilities:

Case 1: Only one qubit gets affected: $\rho \to \rho_1$ (possibly a mixed entangled state).

Case 2: Both the qubits are affected: $\rho \to \rho_2$ (also possibly a mixed entangled state).

Now what can be said about the relative quality of the two states ρ_1 and ρ_2, say, which state is now more useful for teleportation?

Let us try to put things from a more instructive point of view. Assume that, in the beginning only "qubit 1" gets affected by amplitude damping, then $\rho \to \rho_1$. Certainly ρ_1 is of poor quality when compared to ρ. We now allow "qubit 2" also to interact, so that $\rho_1 \to \rho_2$.

Now both the qubits are being affected, one may be tempted to speculate that ρ_2 is definitely worse that ρ_1 in terms of efficiency to perform say, teleportation.

To our surprise we found that the state ρ_2 is *sometimes* "better" than ρ_1 having a higher teleportation fidelity! We use the word "sometimes" because, as will be shown later that this effect is dependent on the strength of the amplitude damping channel affecting the first qubit. This is something very contrary to the common intuition. This not only provides an example when entangled states can become less corrupt when both the qubits are affected instead of one but this result says something more. It says that if an entangled state already corrupted by amplitude damping channel acting on only one qubit, then the lost inseparability can sometimes be partially recovered by allowing for similar dissipative action on the other qubit. Thus we are able to extract, though partially the non local correlations with this prescription. Crudely speaking this sounds some kind of "error correction". By this we mean that the "error" introduced in the nonlocal correlations through interaction of one of the qubits with the environment can be "corrected" partially by allowing the other qubit also to undergo similar dissipative interaction.

Moreover we show that if ρ_1 (the resulting state when the first qubit is affected) is a non teleporting state then it can *always* be made teleporting by allowing the second qubit (which didn't interact with the environment before) to interact with the environment through the amplitude damping channel. This effect of transforming non teleporting states to teleporting ones by allowing dissipative interaction is similar to that obtained by Badziag et al [15].

Importantly all the above consequences depend on the strength of the amplitude damping channel affecting the first qubit. As is well known an amplitude damping may be characterized by a parameter p, interpreted as the probability of transition of a two level atom from its excited state to the ground state in presence of an electromagnetic field (environment). The parameter p also represents the strength of the interaction since larger the value of p, stronger is the interaction, thus affecting more the ability of an entangled state to teleport. We now briefly discuss the action of an amplitude damping channel on a qubit [13].

The amplitude damping channel describes the interaction of a two level atom with the electromagnetic field (environment). Specifically the decay of an excited state of a two level atom by spontaneous emission of a photon in presence of an e.m field is what is modeled by this channel. The Unitary transformation that governs the evolution of the system and the environment is defined by,

$$|0\rangle_A |0\rangle_E \rightarrow |0\rangle_A |0\rangle_E \quad (2)$$

$$|1\rangle_A |0\rangle_E \rightarrow \sqrt{1-p}\,|1\rangle_A |0\rangle_E + \sqrt{p}\,|0\rangle_A |1\rangle_E \quad (3)$$

Physically this implies that if an atom is in an excited state $|1\rangle_A$, the probability that it makes a transition to the ground state $|0\rangle_A$ with the emission of a photon is p. The envi ronment also makes a transition from the "no-photon" state $|0\rangle_E$ to the "one-photon" state $|1\rangle_E$. . Tracing out the environment we obtain the Kraus operators,

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, K_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \quad (4)$$

and it is easy to check that

$$K_0^\dagger K_0 + K_1^\dagger K_1 = 1 \quad (5)$$

implying that the operation is trace preserving.

The density matrix ρ of the quantum system then evolves as

$$\rho \rightarrow \rho'(p) = \Lambda(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger \tag{6}$$

Thus the above equation defines a linear map Λ (a superoperator, describing some physical process, which in this case is the interaction of a two level atom with an e.m. field) which takes a density matrix to another density matrix. It is clear that the amplitude damping channel is *characterized* by the parameter p, the transition probability.

Let us now consider the following situation: Alice and Bob share a maximally entangled state,

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \tag{7}$$

and the density operator corresponding to the above state is $\rho = |\Phi^+\rangle \langle \Phi^+|$. Assume that only Bob's qubit interacts with its local environment via the amplitude damping channel. Such an interaction is described by,

$$\rho \rightarrow \rho'(p) = \Lambda(\rho) = W_0 \rho W_0^\dagger + W_1 \rho W_1^\dagger \tag{8}$$

where $W_i \equiv I \otimes K_i$ are given by,

$$W_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{1-p} \end{bmatrix} ; W_1 = \begin{bmatrix} 0 & \sqrt{p} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{p} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

This interaction turns the maximally entangled state, originally shared by Alice and Bob into a mixed entangled state,

$$\rho'(p) = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & \sqrt{1-p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 \\ \sqrt{1-p} & 0 & 0 & 1-p \end{bmatrix} \tag{10}$$

It may be useful to note that this state is always entangled for $p \neq 1$. The fully entangled fraction of this state is,

$$f(\rho') = \frac{1}{4} \left(1 + \sqrt{1-p}\right)^2 \tag{11}$$

Now for this state to be useful for teleportation we must have $f(\rho') > \frac{1}{2}$ which puts a restriction on the parameter p,

$$p < 2\sqrt{2} - 2 \tag{12}$$

i.e. for all values of $p \geq 2\sqrt{2} - 2$ the state $\rho'(p)$ is not useful for teleportation in the sense that fidelity is not better than what can be achieved by a classical channel.

Let us now allow Alice's qubit also to interact with the environment. It should be kept in mind that the entangled state now shared by Alice and Bob is described by the density operator $\rho'(p)$. We therefore have the following transformation

$$\rho'(p) \rightarrow \rho''(p) = \Lambda'(\rho'(p)) = W_0' \rho'(p) W'^\dagger + W_1' \rho'(p) W_1'^\dagger \tag{13}$$

where $W_i' \equiv K_i \otimes I$ are given by

$$W_0' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{1-p} & 0 \\ 0 & 0 & 0 & \sqrt{1-p} \end{bmatrix}; W_1' = \begin{bmatrix} 0 & 0 & \sqrt{p} & 0 \\ 0 & 0 & 0 & \sqrt{p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

Here the parameter p is taken to be same for the interactions on both sides. We stress that although this is not a necessary assumption but at the moment this assumption simplifies the calculations without changing the consequences that we want to show. A general treatment involving different p's on both sides is given later.

After the interaction defined by (13) the state of Alice and Bob is transformed into the state,

$$\rho''(p) = \frac{1}{2} \begin{bmatrix} 1+p^2 & 0 & 0 & 1-p \\ 0 & p(1-p) & 0 & 0 \\ 0 & 0 & p(1-p) & 0 \\ 1-p & 0 & 0 & (1-p)^2 \end{bmatrix} \tag{15}$$

First note that the above state is always entangled for $p \neq 1$. The fully entangled fraction corresponding to the state $\rho''(p)$ is

$$f\left(\rho''(p)\right) = \frac{1}{2}\left[1 + (1-p)^2\right] \tag{16}$$

One interesting point is to be noted from (16). Observe that the state $\rho''(p)$ is always suitable for teleportation since $f(\rho'') > 1/2$ for all $p (p \neq 1)$. Here it may be worth recalling that for the state $\rho'(p)$ this is not the case always (see Eq. (12)).

Before we proceed to make a qualitative analysis of the two states $\rho'(p)$ and $\rho''(p)$, recall that $\rho'(p)$ is the state that resulted when only Bob's qubit interacted with the environment via amplitude damping channel and $\rho''(p)$ is the resulting state when both the qubits got affected. So if we ask the question:

Which one of the two states is the "better" one in the sense of usefulness to perform, say quantum teleportation?

then it might be tempting to conclude that $\rho'(p)$ should *always* be the qualitatively better state that $\rho''(p)$, but surprisingly it turns out that this is *not* true always. By comparing $f\left(\rho'\right)$ and $f\left(\rho''\right)$ it is easy to obtain that

$$f\left(\rho''\right) > f\left(\rho'\right), \forall p > p_0 \cong 0.806 \tag{17}$$

The value of p_0 is corrected up to three places of decimals.

This means that a state, initially corrupted when only one qubit (say Bob's) interacted with the environment gets "better" when the other qubit (belonging to Alice) is also allowed to interact!

Let us now turn our attention to some other consequences. Earlier we have observed that the state $\rho'(p)$ is not suitable for teleportation when $p \geq 2\sqrt{2} - 2 \cong 0.828$. Since $\rho''(p)$ is always suitable for teleportation and $p_0 < 0.828$, we have here an example of a family of non teleporting states $\rho'(p)(p \geq 2\sqrt{2}-2)$ that are transformed to teleporting ones $\rho''(p)$ by allowing the other qubit (the qubit which didn't interact with the environment before) to interact with its local environment.

Besides these it is also easy to see that the teleporting states $\rho'(p)$, $(0.806 < p < 0.828)$, are also transformed to the states $\rho''(p)$ that have larger fully entangled fraction implying higher teleportation fidelity. Thus the teleportation fidelity of the teleporting states (though not of "good" quality) are seen to be enhanced by this type of dissipative interaction.

This can also be viewed as a way to generate a family of entangled states whose fidelity of teleportation can be enhanced by dissipative interaction with the environment. All states described by the density operator $\rho'(p)$ for $p > 0.806$ belong to this class. Moreover, a subclass of this family are the non-teleporting states that can be transformed to teleporting ones through the same dissipative interaction.

We now treat the problem in a more general way. Now we assume that the disspation stregth is different for both parties. As before, we will assume that first Bob's qubit undergoes dissipative interaction with the environment and the parameter p of the amplitude damping channel will now be denoted by p_b. After that we allow the qubit that belongs to Alice also to interact with the environment via amplitude damping channel and the channel parameter now taken to be different from p_b is denoted by p_a.

The fully entangled fraction $f(\rho')$ of the resulting state $\rho'(p_b)$ after the first interaction is ,

$$f(\rho') = \frac{1}{4}\left(1+\sqrt{1-p_b}\right)^2 \tag{18}$$

This is same as Eq. (11), the only difference is that p is now replaced by p_b. Now when we allow Alice's qubit also to interact with the environment, the state $\rho'(p_b)$ is transformed to $\rho''(p_a, p_b)$. It is not difficult to obtain that the fully entangled fraction corresponding to this new state $\rho''(p_a, p_b)$ is,

$$f(\rho'') = \frac{1}{4}\left[p_a p_b + \left(1+\sqrt{(1-p_a)(1-p_b)}\right)^2\right] \tag{19}$$

Our objective is to obtain the condition such that the inequality

$$f(\rho'') > f(\rho') \tag{20}$$

is satisfied. This means that we are now looking for those values of p_a , such that for a given p_b the previous inequality is satisfied. It turns out that for

$$p_a < \frac{4\left[\sqrt{1-p_b}\,(2p_b-1)-(1-p_b)\right]}{(2p_b-1)^2} = g(p_b) \tag{21}$$

the inequality $f(\rho'') > f(\rho')$ is satisfied.

The important question that remains is: *Does it imply that given any p_b we can always find a suitable range of p_a such that the inequality (20) is satisfied?*

The answer is *no*. This can at once be seen from (21) by observing that $g(3/4) = 0$. This tells us that we can always find some suitable range of p_a such that the inequality (20) is satisfied provided $p_b > 3/4$. Stated more explicitly this means that if and only if had Bob's qubit interacted with the environment via the amplitude damping channel characterized by the parameter $p_b > 3/4$, then the possibility of improving the corrupted state arises by allowing the other qubit to interact in accordance to (21).

Let us now assume $p_b > 3/4$. Thus we have at our disposal a range of possible values of p_a satisfying (21) and the inequality (20). Then another important question remains to be answered: For which value of p_a, $f(\rho'')$ is the maximum? That is, for which value p_a highest fidelity for teleportation can be achieved?

One can easily see that (19) is maximised for

$$p_a = \frac{p_b(4p_b-3)}{(2p_b-1)^2} \tag{22}$$

Here it is interesting to note that (22) immediately gives a lower bound on the value of the parameter p_b, which is $3/4$ and this bound we have obtained before. We can now summarize the results that we have obtained so far.

Alice and Bob initially shared a maximally entangled state. If Bob's qubit gets affected by amplitude damping such that $p_b > 3/4$ then the fidelity of the state can be improved by allowing Alice's qubit to interact with the environment via the amplitude damping channel characterized by the parameter p_a provided,

$$p_a < \frac{4\left[\sqrt{1-p_b}\,(2p_b-1)-(1-p_b)\right]}{(2p_b-1)^2} \tag{23}$$

and the maximum "fully entangled fraction" or the maximum fidelity of teleportation is obtained by substituting the value of p_a given by (22) in the expression (19) and that maximum value of $f(\rho)$ is greater than $1/2$.

Earlier we have found that for $p_b \geq 2\sqrt{2}-2$ the state $\rho'(p_b)$ is not capable of teleportation better than what can be achieved classically. It is now clear that those non teleporting states can be made teleporting by allowing the other qubit to interact with the environment with an appropriate choice of the parameter p_a. It turns out that the allowed range of p_a is,

$$\frac{-2+p_b+2p_b^2-2\sqrt{2}\,(1-p_b)\sqrt{(1-p_b)}}{(1-2p_b)^2} < p_a < \frac{-2+p_b+2p_b^2+2\sqrt{2}\sqrt{(1-p_b)}}{(1-2p_b)^2} \tag{24}$$

In particular the maximum fidelity can be obtained if p_a takes the value given by (22). Now even if the state $\rho'(p_b)$ is a teleporting one (which it is, if $3/4 < p_b < 2\sqrt{2}-2$) still its fidelity can be enhanced by allowing the other qubit to interact with the environment for any value of p_a defined by (23).

While showing the above effects we started with a maximally entangled state which is then subjected to interactions with the local environment. We do not know whether these effects can also be observed if we had chosen the *initial* state to be a general two qubit bipartite entangled state. However calculations [19] with the initial state chosen to be any pure entangled state or a Werner state showed clear evidences of these effects, although the conditions turned out to be complicated.

The possible reasons of the effects that we have discussed in this paper are not immediately clear. It is true that entanglement is playing a major role but more importantly these features seem to be the characteristic of the amplitude damping channel through which the entangled state locally interacted with the environment. It so happens that after one qubit interacts with the environment, the quantum correlations got hidden in a nontrivial way and only sometimes, when the other qubit is also allowed to interact, revealed itself. We emphasize that this is also different from the filtering procedure [16, 17] which involves selection process after measurement thus revealing hidden correlations. However filtering is not a trace preserving operation whereas the actions that we have discussed are entirely trace preserving.

One possible line of thought may be as follows: when only one qubit is affected, the process becomes clearly asymmetric. Subjecting an entangled state into such an asymetric treatment is sometimes seen to be worse than had we allowed for a symmetric one by allowing the other qubit also to interact. When we do so the "symmetry" becomes restored partially thereby revealing the quantum correlations. Thus mixed entangled states are again seem to behave in a curious fashion. Another possible way to view these effects as, something that is similar to "error correction". The "error" introduced in the nonlocal correlations when one qubit interacted with its local environment can be partially "corrected" by allowing the other qubit also to interact.

In conclusion, we have investigated the action of local environment via amplitude damping channel on maximally entangled states. We have shown that entangled states when corrupted through such local interactions only on one side can sometimes become better by allowing the other qubit to undergo similar dissipative interaction. We have also shown that such dissipative interactions that are usually responsible for decoherence and destroying quantum inseparability can also sometimes help in improving the fidelity of quantum teleportation. As a striking consequence of this, transforming non teleporting states to teleporting states is thereby made possible.

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